U.S. DEPARTMENT OF COMMERCE
National Technical Information Service

AD-A035 381

THE CONTINUOUS SPECTRUM OF THE ORR-SOMMERFELD EQUATION

PART I. THE SPECTRUM AND THE EIGENFUNCTIONS

OLD DOMINION UNIVERSITY, NORFOLK, VIRGINIA

SEPTEMBER 1976

Institute of Oceanography
Old Dominion University
Norfolk, Virginia

Technical Report No. 33

The Continuous Spectrum of the Orr-Sommerfeld Equation Part I: The Spectrum and the Eigenfunctions

by
Chester E. Grosch
and
Harold Salwen

Sponsored by
Advanced Research Projects Agency
ARPA Order No. 2924

Monitored by
Office of Naval Research
Under
Contract No. N00014-75-C-0777

December 1976

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER Technical Report No. 33	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE CONTINUOUS SPECTRUM OF THE ORR-SOMMERFELD EQUATION; Part 1: The Spectrum and the Eigenfunctions		5. TYPE OF REPORT & PERIOD COVERED Interim Aug 1, 1975-Jul 31, 1976 6. PERFORMING ORG. REPORT NUMBER Technical Report No. 33
7. AUTHOR(s) Chester E. Grosch Harold Salwen*		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0777
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute of Oceanography Old Dominion University Norfolk, VA 23508		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS ARPA Order No. 2924
11. CONTROLLING OFFICE NAME AND ADDRESS Advanced Research Projects Agency 1400 Wilson Boulevard Arlington, VA 22209		12. REPORT DATE September 1976 13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Office of Naval Research 800 North Quincy Street Arlington, VA 22217		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

* Department of Physics, Stevens Institute of Technology, Hoboken, NJ 07030

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Stability; Transition; Boundary layer; Jet; Wake; Shear flow; Tollmien-Schlichting waves; Spectrum; Continuum; Eigenfunctions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that the Orr-Sommerfeld equation governing the stability of any mean shear flow in an unbounded domain which has finite energy under some Galilean transformation has a continuous spectrum. This result applies to both the temporal and spatial stability problems. Formulae for the location of this continuum in the complex wave speed plane are given. The continuous spectrum and corresponding eigenfunctions are calculated for two sample problems: the Blasius boundary layer and the two-dimensional laminar jet. The nature of the

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

eigenfunctions, which are very different from the Tollmien-Schlichting waves, are discussed. Three mechanisms are proposed by which these continuum modes could cause transition in a shear flow while bypassing the usual linear Tollmien-Schlichting stage.

ARPA Order No. 2924

Program Code No. 5G10

Contract No. N00014-75-C-0777

Short Title of Work - Numerical Transition

Contractor - Old Dominion University Research Foundation

Principal Investigator - Chester E. Grosch [(804) 489-6477]

Scientific Officer - Program Director Fluid Dynamics
Mathematical and Information Sciences Division
Office of Naval Research
800 North Quincy Street
Arlington, VA 22217

Effective Date of Contract - 1 February 1975

Contract Expiration Date - 30 September 1976

Amount of Contract - $97,000

SUMMARY

Understanding the physics and predicting transition from laminar to turbulent flow remains one of the central problems in fluid dynamics. Theoretical and experimental studies have shown that, if free stream disturbances, wall vibrations, etc., are sufficiently small, the appearance and growth of Tollmien-Schlichting waves can dominate the process of boundary layer transition. Recent theoretical studies suggest that the Tollmien-Schlichting waves do not constitute a complete set of solutions of the Orr-Sommerfeld equation and hence cannot be used to represent an arbitrary disturbance. In addition, it has long been recognized that the Tollmien-Schlichting waves are largely confined to a region within two or three boundary layer thicknesses of the solid wall. The excitation of these waves by free stream disturbances which do not penetrate the boundary layer is an unsolved problem.

In this paper it is shown that the Orr-Sommerfeld equation has a continuous spectrum, in addition to the discrete spectrum of the Tollmien-Schlichting waves, for any flow in an unbounded domain such as a boundary layer, jet or wake. Explicit formulae for the location of the continuum in the complex wave speed plane are given. The continuous spectrum and the continuum eigenfunctions are calculated for two sample problems: the Blasius boundary layer and the two-dimensional laminar jet. It is shown that these modes are free stream modes; far from the boundary or jet or wake axis they are vorticity waves. Three mechanisms are suggested by which free stream vorticity could cause transition in a shear flow by exciting the continuum modes. These mechanisms are (1) quasi-steady

distortion of the mean profile; (2) time-dependent instability of the mean flow plus a continuum mode; and (3) the effects of nonparallel flow corrections to the continuum spectrum and eigenfunctions.

Further research is needed to show how a "patch" of vorticity in the free stream (a wave pocket) can be represented in terms of the continuum eigenfunctions; to elucidate the connection between these modes and the forced disturbances studied by others; and to examine the postulated transition mechanisms listed above. We have begun a theoretical study of these problems. It would be desirable if experimentalists would also look for these modes.

1. INTRODUCTION

The study of the stability of laminar flow against externally imposed disturbances and the connection of such instability, if it exists, with the transition from laminar to turbulent flow has been studied for nearly a century and remains one of the central problems in fluid dynamics. Ever since Reynolds' classic experiments (1883) it has been conjectured that transition from laminar to turbulent flow is the result of an instability in the laminar flow. Beginning with Rayleigh (1880), there have been many theoretical studies which have attempted to predict under what conditions small disturbances in the velocity profile would grow or decay.

The main interest in stability calculations centers on the stability of boundary layer flows and related flows such as jets, wakes, free shear flows; Couette and particularly Poiseuille flow have had the status of "canonical" problems. Tollmien (1931), Schlichting (1951) and Lin (1955) made notable advances in the analytical calculation of the stability of these parallel shear flows to two-dimensional wavelike disturbances, the Tollmien-Schlichting waves (hereafter called T-S waves). The restriction to two-dimensional waves was justified by a result of Squire (1933), who showed that if an oblique wave is unstable at some Reynolds number, then a two-dimensional T-S wave is unstable at a smaller Reynolds number. Thus, in order to find the critical Reynolds number it suffices to consider only two-dimensional T-S waves.

Although the theory was highly developed, it was, prior to the early 1940's, generally regarded as irrelevant to the actual mechanics of transition because T-S waves had never been observed experimentally, and it appeared that any disturbance, whether in the free stream, in the boundary layer, or on the boundary, could cause a transition if it were large enough or if the Reynolds number were high enough.

The experimental studies of Schubauer and Skramstad (1941, 1947, 1948) first showed that, if the free stream disturbances, wall vibrations, etc., were sufficiently small, the appearance and growth of T-S waves could play a dominant role in boundary layer transition. Schubauer and Skramstad found generally good agreement between the predictions of linear stability theory and the experimental results, not only for the shape of the neutral stability curve and the phase speeds of the T-S waves, but also for the shape of the T-S waves as a function of distance from the boundary.

While the work of Schubauer and Skramstad showed the essential correctness of linear stability theory, the calculations of the detailed stability characteristics for a particular profile remained quite delicate in that the results obtained were sensitive to the detailed shape of the profiles and the approximations used in the asymptotic analysis. It is only the development of sophisticated numerical methods and the high speed computers to implement them (Kaplan, 1964; Gallagher and Mercer, 1964; Landahl, 1966; Radbill and Van Driest, 1966; Wazzan, Okamura and Smith, 1968; Lessen et al., 1968; Grosch and Salwen, 1969; Orszag, 1971; Salwen and Grosch, 1972) that has made these calculations relatively routine.

It is usually assumed that the eigenvalue problem of linear stability theory has an infinite set of discrete eigenvalues and a corresponding infinite set of eigenfunctions. Assuming the completeness of this set of eigenfunctions, an arbitrary, initial disturbance which satisfies the boundary conditions can be expanded in terms of them. We know of no general proof of this completeness "theorem."

Almost all stability calculations have been concerned with finding the first or least stable eigenvalue. It should be noted that, if expansion

(spectral) techniques (Grosch and Salwen, 1969; Orszag, 1971; Salwen and Grosch, 1972) are used to solve the eigenvalue problem in a finite domain, a number of the higher eigenvalues are found numerically along with the first. If a sufficient number of terms is used in the expansion, these numerically calculated eigenvalues will overlap the asymptotic (high order) distribution of eigenvalues, for which analytic formulae may be derived, and the complete spectrum may be found. Apart from any difficulties in the analysis or calculation, this concentration on the first eigenvalue may be due to the facts that 1) experiments suggest that there is only a single unstable mode and 2) for those flows for which the higher modes have been calculated, plane Couette (Gallagher and Mercer, 1964), plane Poiseuille (Grosch and Salwen, 1969; Orszag, 1971), pipe Poiseuille (Lessen et al., 1968; Salwen and Grosch, 1972), these higher modes have been found to be highly damped.

Very recently Jordinson (1971) and Mack (1975), using different numerical methods, have calculated the higher eigenvalues of the Orr-Sommerfeld equation for Blasius flow. Jordinson calculated eigenvalues for the spatially and temperally growing or decaying modes for a single Reynolds number, and a single wave number (temporal problem) or a single frequency (spatial problem). Mack calculated the eigenvalues for a number of different values of the wave number and Reynolds number for the temporal stability problem. While there is some disagreement in the number and location of the eigenvalues found by Jordinson and Mack, they both agree that at any Reynolds number there are only a finite, and small, number of discrete eigenvalues. A finite set of eigenfunctions cannot be a complete set. How, then, can an arbitrary initial disturbance be expanded? Both Jordinson and Mack suggest that there is a continuous spectrum.

We know of no general results concerning the spectrum of the Orr-Sommerfeld equation for an arbitrary parallel shear flow in an unbounded domain. For the Orr-Sommerfeld equation in a bounded domain or in the case of the inviscid stability problem, for which the governing equation is the Rayleigh equation, a number of specific problems have been studied in detail, and some general results concerning the spectrum are available.

Much is known about the spectrum of the inviscid stability problem (the Rayleigh problem). Howard's circle theorem (1961) states that all eigenvalues of the Rayleigh problem lie in a circle in the complex plane whose diameter is the range of undisturbed velocity. Case (1960) has studied the spectrum of inviscid plane Couette flow and shown that there are no discrete eigenvalues; there is only a continuum. In the same paper Case generalized this result and showed that for any mean velocity profile in a finite domain, the Rayleigh problem always has a continuous spectrum in addition to the discrete spectrum, if one exists. Rosencrans and Sattinger (1966) and Sattinger (1967) have proved, under fairly general conditions on the velocity profile of the mean flow, that the Rayleigh problem in a finite domain has only a finite number of discrete eigenvalues. Plane Couette and Poiseuille flows and, it appears, most other physically reasonable flows in a finite domain satisfy the conditions of this theorem.

Case (1961) and Lin (1961) have examined the connection between the spectrum of the inviscid problem and that of the viscous problem for a finite domain. Case has shown that, if the initial conditions are independent of the viscosity (Reynolds number), the solution of the viscous problem approaches the solution of the inviscid problem as the viscosity goes to zero and, hence, has a continuous spectrum. Lin has pointed out that, for the

viscous problem, the initial conditions depend in general on the viscosity and, therefore, Case's theorem does not apply. Lin has further shown that the Orr-Sommerfeld equation has only discrete eigenvalues in a bounded domain.

It has been proved that the Orr-Sommerfeld equation has a complete set of eigenfunctions for the specific cases of plane Couette flow (Haupt, 1912) and plane Poiseuille flow (Schensted, 1960). Di Prima and Habelter (1969) have proved a completeness theorem for a class of nonself-adjoint eigenvalue problems in a bounded domain. Using this theorem they showed that the Benard problem (linear stability of a layer of fluid heated from below), the Taylor problem (linear stability of the flow between rotating concentric cylinders), and the Orr-Sommerfeld equation for any flow in a bounded domain have a complete set of eigenfunctions.

None of these results apply to the Orr-Sommerfeld equation if the domain is infinite as it is for, say, Blasius flow. The possibilities for the spectrum of the Orr-Sommerfeld equation in an infinite domain are: (a) there is an infinite set of discrete eigenvalues without a continuum; (b) there is an infinite set of discrete eigenvalues with a continuum; (c) there is a finite number of eigenvalues without a continuum; or (d) there is a finite number of discrete eigenvalues with a continuum.

In this paper we will <u>not</u> be concerned with whether or not the discrete spectrum of the Orr-Sommerfeld equation is finite in an infinite domain. We will consider the existence of the continuous spectrum of the Orr-Sommerfeld equation for both the temporal and spatial problem. In section 2, it is shown that the Orr-Sommerfeld equation has a continuous spectrum for a wide class of shear flows in an infinite domain. Formulae for the continuous spectrum in terms of the wave number or frequency and the Reynolds number

are given for these flows. Detailed calculations of the continuous spectrum and examples of the continuum eigenfunctions for two specific flows, the Blasius boundary layer and the two-dimensional laminar jet, are given in section 3. Also included in section 3 is a physical interpretation of the modes. Section 4 contains some speculation on mechanisms by which these modes could lead to transition. Finally, section 5 is a brief summary of our results.

In a later paper (Part II) we will consider detailed representation of particular disturbances in terms of these eigenfunctions and their spatial and temporal evolution. We also plan to study at some later time the mechanisms (suggested in section 4) by which these modes might lead to transition.

2. GENERAL ANALYSIS

2.1. *Formulation*

The basic flow is a parallel shear flow (U (y), 0, 0), in a Cartesian coordinate system (x, y, z). We consider infinitesimal two-dimensional disturbances to this flow. The stream function of the disturbance is assumed to be of the form

$$\psi\ (x, y, t) = \phi\ (y)\ e^{i\alpha(x-ct)} \quad . \qquad (1)$$

As is well known, with these assumptions the linearized Navier-Stokes equations reduce to the Orr-Sommerfeld equation:

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 \phi = i\alpha\ R\left[\left(U - c\right)\left(\frac{d^2}{dy^2} - \alpha^2\right) - \left(\frac{d^2U}{dy^2}\right)\right] \phi \qquad (2)$$

All variables are dimensionless; the length scale is L, the velocity scale is U_o and the time scale L/U_o. As usual, α is the wave number, c the

phase speed, and R the Reynolds number, $U_o L/\nu$, with ν the kinematic viscosity. Equation (2) is to be solved with suitable boundary conditions (discussed below). In the temporal stability problem α is real and c is complex, so that the flow is unstable if $\mathrm{Im}\,(c) > 0$. For the spatial stability problem,

$$\omega = \alpha c \tag{3}$$

is real and α is complex; the flow is unstable if $\mathrm{Im}\,(\alpha) < 0$.

The disturbance, equation (1), is just a single eigenfunction of the stability problem. If, as is the case for the temporal stability problem of plane Poiseuille flow, the Orr-Sommerfeld equation has an infinite set of discrete eigenvalues $\{c_n\}$ and a corresponding complete set of eigenfunctions $\{\phi_n\,(y)\}$, the most general solution of the linearized Navier-Stokes equations has, at fixed R, a stream function of the form,

$$\begin{aligned} \Psi\,(x, y, t) &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n\,(\alpha)\,\psi_n\,(x, y, t; \alpha)\,d\alpha \\ &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n\,(\alpha)\,\phi_n\,(y; \alpha)\,e^{i\alpha(x-c_n(\alpha)t)}\,d\alpha\,, \end{aligned} \tag{4}$$

where the $\{A_n\,(\alpha)\}$ are "Fourier" amplitudes. A physical disturbance in the flow will, in general, excite a mixture of modes, and this, for the case of temporal stability, corresponds to specifying $\Psi\,(x, y, o)$. If any mode $\psi_n\,(x, y, t; \alpha)$ grows in time, then the initial disturbance, or rather part of it, grows in time.

In order to generate only a single mode with a given wave number α_o, $\Psi\,(x, y, o)$ must be of the form

$$\Psi\ (x,\ y,\ o) = A\ \phi_n\ (y;\ \alpha_o)\ e^{i\alpha_o x}\ , \tag{5}$$

with A some arbitrary amplitude. It is difficult, at best, to conceive of a physical experiment which could generate an initial disturbance exactly of the form of (5).

Actually the temporal stability problem does not represent the controlled experiments that are carried out in shear flows. These experiments involve the generation of a disturbance of fixed frequency at some fixed x position, say x_o, and observation of the spatial growth or decay of the disturbance, in this case

$$\Psi\ (x,\ y,\ t) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n\ (\omega)\ \phi_n\ (y;\ \omega)\ e^{i(\alpha_n x-\omega t)}\ d\omega\ . \tag{6}$$

In principle $\Psi\ (x_o,\ y,\ t)$, which is of the form

$$\Psi\ (x_o,\ y,\ t) = F\ (y)\ e^{i\omega_o t}\ , \tag{7}$$

is known, and so the $\{A_n\ (\omega)\}$ can be found. If, as appears to be the case for the Blasius flow, the higher T-S modes are highly damped compared to the fundamental T-S mode, they decay rapidly with increasing x, and some distance downstream from the disturbance generator only the fundamental mode is observed.

Clearly the T-S modes are only a mathematical device for representing physical disturbances. It is only an accident that the higher T-S modes are highly damped so that some distance downstream from a disturbance generator the pure, fundamental T-S wave can be observed, even though a pure, fundamental T-S wave is not generated by the physical disturbance generator. If the

Orr-Sommerfeld equation has eigenfunctions other than the T-S modes, they too must be regarded as mathematical representations of portions of physically realizable disturbances.

For a general shear flow it is conceivable that the Orr-Sommerfeld equation might not have an infinite set of discrete eigenvalues and so could not possibly have a complete set of corresponding eigenfunctions or that, even if it had an infinite set of discrete eigenvalues, the corresponding infinite set of eigenfunctions might not be complete. In either of these cases completeness could require the inclusion of "improper eigenfunctions" (Friedman, 1956) corresponding to a continuous spectrum. If this were the case, then the stream function could not, in general, be represented as a sum over only the discrete modes, as in equation (4); the contribution from the continuum must be included, i.e., for the temporal problem

$$\Psi = \int_{-\infty}^{\infty} \left\{ \sum_n A_n(\alpha)\, \phi_n(y;\alpha)\, e^{i\alpha(x-c_n t)} + \int B(\alpha, c)\, \phi(y;\alpha, c)\, e^{i\alpha(x-ct)}\, dc \right\} d\alpha\,, \tag{8}$$

where the integral in the second term is taken over the continuum of c.

This latter case is quite common in other branches of physics. Consider, for example, the nonrelativistic Schrödinger equation (Landau and Lifschitz, 1958) for an electron in a potential. If the potential is that of Hydrogen, it is easily shown that there is an infinite set of discrete eigenvalues and corresponding eigenfunctions which are not complete. Completeness requires the inclusion of the continuous spectrum and "improper" eigenfunctions. If the potential is that of the Deuteron, there is only one discrete eigenvalue, and completeness requires the inclusion of the continuum modes (Bethe, 1947).

In both these cases the discrete eigenvalues are the energy levels of the bound states of the electron, and a point on the continuum is the energy of an electron which is scattered by the potential.

A trivial example may serve to illustrate the essential point of this argument. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad . \tag{9}$$

If we look for solutions of the form

$$u\ (x,\ t) = f(x)\ e^{i\omega t} \quad , \tag{10}$$

then $f(x)$ is a solution of

$$\frac{d^2 f}{dx^2} + \omega^2 f = 0 \ . \tag{11}$$

If the boundary conditions for equation (9) are

$$u\ (o,\ t) = u\ (1,\ t) = 0 \quad , \tag{12}$$

then there is an infinite set of discrete eigenvalues $\{\omega_n\}$ and eigenfunctions $\{f_n\ (x)\}$

$$\omega_n = n\Pi \quad , \tag{13}$$

$$f_n\ (x) = \frac{1}{\sqrt{2}} \sin\ (n\Pi x) \ , \qquad n = 1,\ 2,\ 3,\ .\ .\ .\ .\ , \tag{14}$$

and these eigenfunctions are orthonormal and are a complete set. If we consider the infinite domain $0 \leq x < \infty$, instead of the finite domain

$0 \leq x \leq 1$, and impose the boundary conditions

$$u\,(o,\,t) = 0\ , \tag{15a}$$

$$u\,(x,\,t) \to 0 \quad \text{as} \quad x \to \infty\ , \tag{15b}$$

it is obvious that there are a finite number of discrete eigenvalues; in fact, there are exactly zero eigenvalues. If, however, the boundary condition (15b) is relaxed to

$$u\,(x,\,t) \quad \text{is bounded as} \quad x \to \infty\ , \tag{15c}$$

then the spectrum is a continuum,

$$\omega \ \ \text{real and} \ \ \omega \geq 0\ ,$$

and the "improper" eigenfunctions are

$$f(x;\,\omega) = \frac{1}{\sqrt{2\Pi}} \sin \omega x\ . \tag{16}$$

These eigenfunctions are improper in the sense that they are not square integrable in $[0, \infty)$, but, as written, they are δ function normalized, i.e.,

$$\int_0^\infty f(x;\,\omega)\, f(x;\,\omega')\, dx = \delta\,(\omega - \omega')\ . \tag{17}$$

This example is, of course, trivial. The Fourier sine series on [0, 1] has been replaced by the Fourier sine integral on $[0, \infty)$. The fact that the continuum eigenfunctions are improper (not square integrable in $[0, \infty)$) is so familiar as to be accepted without comment. The fact that $f(x;\,\omega)$ is not a physically realizable mode but only a mathematical representation of a

portion of a wave packet, which is physically realizable, is a familiar concept.

It will be shown below that the Orr-Sommerfeld equation for a wide class of shear flows in an infinite domain always has a continuous spectrum and that the corresponding continuum or "improper" eigenfunctions are δ function normalizable. The first step in showing this requires an examination of the boundary conditions.

2.2. Boundary Conditions for the Orr-Sommerfeld Equation

The Orr-Sommerfeld equation (2) is to be solved with suitable boundary conditions in $[0, \infty)$. First consider the boundary conditions at $y = 0$.

If the flow is a boundary layer, i.e., there is a solid wall at $y = 0$, then the velocity components (u, v, o) of the disturbance must vanish at $y = 0$. Since

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = - \frac{\partial \psi}{\partial x} \tag{18}$$

two of the boundary conditions for a boundary layer are then

$$\phi = \frac{d\phi}{dy} = 0 \quad \text{at} \quad y = 0 \; . \tag{19}$$

If the flow is an unbounded shear flow such as a jet, wake or free shear layer, ϕ can be written as a sum of symmetric and antisymmetric modes (symmetry about $y = 0$). If $U(y)$ is symmetric about $y = 0$, as is usual for a jet or wake, the symmetric and antisymmetric modes are uncoupled solutions of equation (2). However, if $U(y)$ is antisymmetric, a free shear layer for example, the symmetric and antisymmetric modes are coupled. For ϕ symmetric about $y = 0$, the boundary conditions are

$$\frac{d\phi}{dy} = \frac{d^3\phi}{dy^3} = 0 \quad \text{at} \quad y = 0 \ , \tag{20}$$

while the boundary conditions for an antisymmetric mode are

$$\phi = \frac{d^2\phi}{dy^2} = 0 \quad \text{at} \quad y = 0 \ . \tag{21}$$

What boundary conditions are to be applied at infinity? It is almost universal to require that u and v vanish at infinity and hence that

$$\phi \ , \ \frac{d\phi}{dy} \to 0 \quad \text{as} \quad y \to \infty \ . \tag{22}$$

This condition insures, of course, that

$$\int_0^\infty \phi^* \, \phi \, dy < \infty \ , \tag{23}$$

with the star denoting complex conjugate, i.e., ϕ , is in L_2 . This boundary condition will not be used in this paper. Instead the weaker condition

$$\phi \ , \frac{d\phi}{dy} \quad \text{are bounded as} \quad y \to \infty \tag{24}$$

will be imposed.

Clearly if u , $v \to 0$ as $y \to \infty$ they are also bounded as $y \to \infty$. But, if there are modes that are bounded as $y \to \infty$ which do not satisfy (22) (u, $v \to 0$ as $y \to \infty$), then they are "improper" eigenfunctions and must be regarded as a mathematical device for representing a class of physical disturbances. It will be shown below that the eigenvalues of the modes which satisfy (24) but not (22) form a continuum, and the sum over modes in (4) and

(6) must be replaced by a sum over the discrete modes and an integral over the continuum as in equation (8).

2.3. The Continuous Spectrum

2.3.1. The Temporal Stability Problem. Equation (2) is a fourth order linear differential equation and therefore has four linearly independent solutions: $\phi_j(y)$, $j = 1, 2, 3, 4$. The asymptotic (i.e., $y \to \infty$) form of these solutions can be found by considering equation (2) as $y \to \infty$. For the class of shear flows we are considering

$$U(y) \to U_1 \quad \text{(a constant)} ,$$

$$U'(y) \to 0 , \tag{25}$$

and

$$U''(y) \to 0 ,$$

as $y \to \infty$. If $U(y)$ does not approach a constant as $y \to \infty$, then there is no Galilean transformation under which the flow has finite energy. The constant U_1 is unity for a boundary layer, a wake or a shear layer and is zero for a jet.

Then, as $y \to \infty$, equation (2) becomes

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 \phi = i\alpha R \left[\left(U_1 - c\right)\left(\frac{d^2}{dy^2} - \alpha^2\right)\right] \phi , \tag{26}$$

a fourth order differential equation with constant coefficients. The four independent solutions of the Orr-Sommerfeld equation are then asymptotic to the solutions of equation (26), i.e.,

$$\phi_j\ (y) \sim e^{\lambda_j y} , \tag{27}$$

$$\lambda_1 = - Q^{1/2} \tag{28a}$$

$$\lambda_2 = + Q^{1/2} \tag{28b}$$

$$\lambda_3 = - \alpha \tag{28c}$$

$$\lambda_4 = + \alpha \tag{28d}$$

$$Q = i\alpha\ R\ (U_1 - c) + \alpha^2 . \tag{29}$$

ϕ_1 and ϕ_2 are the "viscous" solutions while ϕ_3 and ϕ_4 are "inviscid" solutions. For the temporal stability problem and a general complex c, λ_1 has a negative real part and λ_2 has a positive real part, so only ϕ_1 and ϕ_3 satisfy the boundary conditions, equation (24),

$$\phi\ ,\ \frac{d\phi}{dy} \text{ bounded as } y \to \infty .$$

A linear combination of ϕ_1 and ϕ_3, the decaying viscous and inviscid solutions, also satisfy the more stringent boundary conditions, equation (22).

The Orr-Sommerfeld equation, (2), can then be solved for ϕ_1 and ϕ_3. Applying the boundary conditions at $y = 0$, the eigenvalue problem is reduced to finding a value of c for which a linear combination of ϕ_1 and ϕ_3 will satisfy two boundary conditions, equation (19), (20), or (21) at $y = 0$.

Note that all these solutions satisfy equation (22) as well as equation (24), i.e., they are in L_2. However, there is another class of solutions which satisfy equation (24) but not equation (22). To see the asymptotic $(y \to \infty)$ form of these solutions, assume that Q is real and negative, i.e.,

$$\mathrm{Re}\ (Q) = \alpha R c_i + \alpha^2 < 0\ , \tag{30}$$

$$\mathrm{Im}\ (Q) = \alpha R\ (U_1 - c_r) = 0\ . \tag{31}$$

From equation (30),

$$c_i < -\ \alpha/R\ , \tag{32}$$

which is written, for convenience,

$$c_i = -\ (1 + k^2)\ \alpha/R\ , \quad k \text{ real and nonzero}\ . \tag{33}$$

Since $\alpha R \neq 0$, equation (31) gives

$$c_r = U_1\ . \tag{34}$$

The trivial solution, $\alpha = 0$, implies $\phi = 0$.

Now with

$$c = U_1 - i\ (1 + k^2)\ \alpha/R\ , \tag{35}$$

$$Q = -\ k^2\ \alpha^2\ , \tag{36}$$

$$\lambda_1 = -\ ik\alpha\ , \tag{37a}$$

and

$$\lambda_2 = +\ ik\alpha\ . \tag{37b}$$

Therefore, both viscous solutions ϕ_1 and ϕ_2 , as well as ϕ_3 , the decaying inviscid solution, satisfy the boundary condition, equation (24), as $y \to \infty$, and

$$\phi = A\,\phi_1 + B\,\phi_2 + C\,\phi_3\,,$$

$$\sim Ae^{-ik\alpha y} + Be^{ik\alpha y} + Ce^{-\alpha y}\,, \quad y \to \infty\,. \tag{38}$$

The independent solutions, ϕ_1 , ϕ_2 , and ϕ_3 , to equation (2) are found, and the boundary conditions at $y = 0$, equation (19), (20), or (21) are applied. Because there are three arbitrary constants and only two conditions to be satisfied, the boundary condition at $y = 0$ can always be satisfied.

The temporal stability problem for the Orr-Sommerfeld equation has, therefore, a continuous spectrum along the line in the complex c plane:

$$c = U_1 - i\,(1 + k^2)\,\alpha/R\,, \tag{39}$$

for arbitrary real positive k . Now, $U_1 = 1$ for a (suitably normalized) boundary layer, wake and shear layer and $U_1 = 0$ for a jet. All of these continuum modes are damped ($c_i < 0$); however, the damping rate,

$$\alpha c_i = -\,(1 + k^2)\,\alpha^2/R\,, \tag{40}$$

is quite small, at least for small k , because $\alpha^2/R << 1$ in most situations. The continuum eigenfunctions are the form of equation (38), where two of the three constants are determined from the boundary conditions at $y = 0$ and the third is arbitrary.

When $k = 0$,

$$\lambda_1 = \lambda_2 = 0 \tag{41}$$

and

$$\phi_1 \sim 1\,, \tag{42a}$$

$$\phi_2 \sim y \,, \tag{42b}$$

and ϕ is once again a linear combination of ϕ_1 and ϕ_3. In general this value of c will not be an eigenvalue.

2.3.2. The Spatial Stability Problem. The case of spatial stability can be treated in an exactly similar way. In this case ω is real and α is complex and we require that

$$\mathrm{Re}\,(Q) = \alpha_r^{\,2} - \alpha_i^{\,2} - RU_1\,\alpha_i < 0 \,, \tag{43}$$

and

$$\mathrm{Im}\,(Q) = 2\alpha_r\,\alpha_i + RU_1\,\alpha_r - R\omega = 0 \,. \tag{44}$$

From equation (44),

$$\alpha_i = \frac{1}{2} R \left(\frac{\omega}{\alpha_r} - U_1\right) . \tag{45}$$

Let the left-hand side of the inequality, equation (43) be set equal to $-\,k^2\,R^2/4$, k real and nonzero.

Then, substituting for α_i from equation (45), it is found that

$$\alpha_r = \left\{\frac{1}{2}\,[\sqrt{\beta^2 + \omega^2\,R^2} - \beta]\right\}^{1/2} \tag{46}$$

with

$$\beta = R^2\,(U_1^{\,2} + k^2)/4 \,; \tag{47}$$

therefore, the continuous spectrum for the spatial stability problem lies on the curve given parametrically by equations (45) through (47). It can be shown

that the positive square root must be taken in equation (46). If the negative sign is taken, equation (43) implies that $\alpha_r^2 < 0$, which is impossible. It can further be shown that both $\alpha_r > 0$ and $\alpha_i > 0$. Since $\alpha_i > 0$, all of the continuum modes are damped for the spatial stability problem.

It is interesting to examine a few special cases. In most cases of interest $\omega/R << 1$. For a boundary layer, wake or free shear layer, $U_1 = 1$, and it is easily shown from equations (45) through (47) that, with $\omega/R << 1$ and $k \to 0$,

$$\alpha_r \to \omega \quad , \tag{48}$$

$$\alpha_i \to R \left[\frac{1}{4} k^2 + (\omega/R)^2\right] \quad , \tag{49}$$

and

$$c \simeq 1 - i\left[\frac{1}{4} k^2 + (\omega/R)^2\right]/(\omega/R) \quad ; \tag{50}$$

and for $k \to \infty$,

$$\alpha_r \to \omega/k \quad , \tag{51}$$

$$\alpha_i \to kR/2 \quad , \tag{52}$$

$$c \simeq 4(\omega/R)^2 k^{-3} - i2 (\omega/R) k^{-1} \tag{53}$$

In all cases c_r, the phase speed, is less than or equal to the free stream speed. The least damped modes are those for which $c_r \simeq 1$ $(k \to 0)$ or

$c_r \simeq 0$ $(k \to \infty)$. The former are progressive waves moving at nearly the free stream speed, and the latter are standing waves.

$U_1 = 0$ for a jet and it is easily seen that, with $\omega/R \ll 1$, as $k \to 0$,

$$\alpha \to \sqrt{\omega R/2}\ (1 + i) , \tag{54}$$

$$c \simeq \sqrt{\omega/2R}\ (1 - i) ; \tag{55}$$

while as $k \to \infty$,

$$\alpha \to \omega k^{-1} + iRk/2 , \tag{56}$$

$$c \simeq 4\ (\omega/R)^2\ k^{-3} - i2(\omega/R)k^{-1} . \tag{57}$$

In the limit $k \to \infty$, the same results are obtained for α and c whether $U_1 = 0$ or $U_1 = 1$. The reason for this is that the limiting process is the same, i.e., $\lim U_1/k \to 0$.

3. APPLICATION TO SPECIFIC SHEAR FLOWS

3.1 The Blasius Boundary Layer

The flow is that past a semi-infinite flat plate, and the velocity profile $U(y)$ is the Blasius boundary layer profile (Batchelor, 1967). The velocity and length scales chosen in this case are U_0 , the free stream speed, and

$$L(x) = \sqrt{\nu x/U_0} , \tag{58}$$

where x is the distance from the leading edge. With this choice of length scale the momentum, δ^* , and boundary layer, δ , thickness are 1.72 $L(x)$ and 5.04 $L(x)$, respectively. Since $U(y) \to 1$ as $y \to \infty$, $U_1 = 1$ and for the temporal stability problem the continuous spectrum lies along

$$c_T = 1 - i\ (1 + k^2)\ \alpha/R , \tag{59}$$

in the complex wave number plane, while for the spatial stability problem the continuous spectrum lies along the curve in the complex wave number plane given parametrically by

$$c_s = \omega\alpha^*/|\alpha|^2 , \tag{60}$$

$$\alpha_r = \left\{\frac{1}{2}\left[\sqrt{\beta^2 + \omega^2 R^2} - \beta\right]\right\}^{1/2} , \tag{61}$$

$$\alpha_i = \frac{1}{2} R \left(\omega/\alpha_r - 1\right) , \tag{62}$$

and

$$\beta = R^2 \left(1 + k^2\right)/4 . \tag{63}$$

Jordinson (1971) calculated the discrete temporal and spatial eigenvalues of the Orr-Sommerfeld equation for Blasius flow for the values:[1]

temporal stability

$R = 580.0$, $\alpha = 0.179$,

and

spatial stability

$R = 580.0$, $\omega = 0.0652$,

while Mack (1976) calculated the discrete eigenvalues of the temporal stability problem for a number of values of α and R including $\alpha = 0.179$, and $R = 580.0$.

[1] The values given here differ from those in Jordinson's paper because he used δ^* as the length scale, instead of L , as given in equation (58). Mack used L as the length scale.

These cases lie within the neutral stability curves in the (α, R) plane; i.e., there is an unstable T-S mode for both the temporal and spatial stability problem. They lie near the top of the unstable region about halfway between branch I and branch II (Obremski et al., 1969).

Figure 1 shows the complex c plane. The continuous spectra for both the temporal problem ($\alpha = 0.179$, $R = 580.0$) and the spatial problem ($\omega = 0.0652$, $R = 580.0$) are shown as well as the T-S eigenvalues computed by Jordinson and Mack.[2] There is substantial disagreement between Jordinson and Mack as to the number and location of the discrete T-S eigenvalues. None of the T-S modes found by Mack lie on the continuous spectrum.

Most of the decaying spatial modes found by Jordinson appear to lie on a curve roughly parallel to the spatial continuous spectrum. It is difficult to see how Jordinson could have found any points on the spatial continuum since his boundary condition as $y \to \infty$ (equation (3) of his paper) is equivalent to requiring that ϕ and its derivatives vanish as $y \to \infty$, and the continuum modes do not approach zero but only remain bounded as $y \to \infty$.

We have calculated the continuum eigenfunctions of the temporal problem, for the values of (α, R) given above, at a large number of points on the continuous spectrum. The numerical method is very similar to that used by Mack. For example, consider the temporal problem at a given α and R. A value of k is chosen, and from equation (59) a point on the spectrum is determined. The upper boundary condition is applied at some large y, say y_0, where equation (25) is satisfied to at least one part in 10^8. A value

[2] Jordinson's computed values were estimated from figure 1 of his paper, and Mack's were taken from table 1a of his paper.

of $y_0 = 9$ is sufficient at $\alpha = 0.179$, $R = 580$; then at y_0 :

$$\phi_1 = e^{-ik\alpha y_0} , \tag{64}$$

$$\phi_2 = e^{ik\alpha y_0} , \tag{65}$$

and

$$\phi_3 = e^{-\alpha y_0} , \tag{66}$$

From these equations $\phi_j{}'(y_0)$, $\phi_j{}''(y_0)$, and $\phi_j{}'''(y_0)$ are determined. The Orr-Sommerfeld equation, (2), is then numerically integrated, using a fourth order Runge-Kutta method, to $y = 0$. A linear combination of these three independent solutions is then chosen to satisfy the boundary conditions, in this case equation (19). This leaves one of the three constants undetermined. Choosing this constant effectively determines the normalization. The normalization we have chosen is to set A, the coefficient of ϕ_1 equal to one.

The nature of these eigenfunctions can be seen by considering them in the far field, i.e., as $y \to \infty$. For any k , the continuum stream function is, as $y \to \infty$,

$$\psi \sim e^{i\alpha(x-ky-ct)} + Be^{i\alpha(x+ky-ct)} + Ce^{-\alpha y}\, e^{i\alpha(x-ct)}$$
$$= e^{-(1+k^2)\alpha^2 t/R} \left\{ e^{i\alpha(x-ky-t)} + Be^{i\alpha(x+ky-t)} + Ce^{-\alpha y}\, e^{i\alpha(x-t)} \right\} . \tag{67}$$

The amplitude of the stream function is decaying exponentially in time with a decay time of $R/[\alpha^2\,(1 + k^2)]$. Apart from this decaying amplitude, the first term in the bracket is a progressive wave, with an amplitude of 1, traveling toward the solid boundary. The magnitude of the wave number is

$\alpha \sqrt{1 + k^2}$, and the propagation vector lies at an angle, $- \Theta$, with respect to the x-axis, where

$$\Theta = \tan^{-1} (k) \quad . \tag{68}$$

The second term is an outgoing wave with complex amplitude B. The wave number is the same magnitude as the incoming wave, and the propagation vector is at the angle $+ \Theta$ with respect to the x-axis. The third term is a "wall wave" with complex amplitude C, which decays outwards from the wall. This wave is propagating parallel to the wall with wave number α .

In the language of scattering, these continuum modes consist of an incoming wave (amplitude = 1 and angle of incidence = Θ) which interacts with the boundary layer and reflects as an outgoing wave (complex amplitude = B and angle of reflection = Θ). A "wall wave" (complex amplitude = C) is generated which, in combination with the incident and reflected waves, satisfies the boundary conditions at $y = 0$. The complex amplitudes B and C are, of course, determined by the detailed shape of the velocity profile through the Orr-Sommerfeld equations and the boundary conditions at $y = 0$. For a fixed α and R , B and C also vary with k .

The magnitude and phase of the complex amplitudes B and C are shown in figure 2 as a function of Θ , the angle of incidence of the incoming wave, for the temporal stability continuum at $\alpha = 0.179$, $R = 580.0$. The magnitudes of B and C are plotted in figure 2a, and the phases of B and C are plotted in figure 2b. Both the magnitudes and phases show somewhat complicated variation with θ .

As $\Theta \to 0$, i.e., at grazing incidence $|B| \to 1$, phase $(B) \to \Pi$, $|C| \to 0$, and phase $(C) \to - 99°$. As Θ increases, $|B|$ and phase (B)

both decrease slowly up to $\theta \simeq 65°$. In this same range $|C|$ increases and the phase (C) decreases. The maximum of $|C|$ occurs at $\theta \simeq 65°$, where $|C| \simeq 3.75$ and phase $(C) \simeq -3\Pi/2$. For larger θ , $|C|$ decreases very rapidly, and both phase (B) and phase (C) undergo rapid variation. The variations of phase (B) and phase (C) are only shown up to $\theta \simeq 85°$. The variation of the phase with θ is so rapid for larger angles of incidence that it is not possible to show it on this figure.

The real and imaginary parts of ϕ are shown in figure 3 as a function of y for the case $\alpha = 0.179$, $R = 580.0$, and $k = 2.0$ $(\theta = 63.43°)$. It can be seen that the disturbance is very small within the lower part of the boundary layer (the top of the boundary layer is at $y = 5.04$). This was found to be true for all θ . These continuum modes are essentially free stream modes and do not penetrate very far into the boundary layer.

Contours of the stream function, ψ , in the far field for this same case are shown in figure 4. The region shown is $0 \leq \alpha x \leq 4\Pi$ and $5\Pi < \alpha y \leq 7\Pi$. It can be seen that these continuum modes are a doubly periodic array of vortices in the free stream. The ratio of the wavelength in the x-direction to that in the y-direction is k , in this case 2.0. Because $|A| \neq |B|$, the axes are tilted and the vortices are distorted. The vortex array is moving with the free stream, i.e., has a phase speed of 1.0, and the vortex strength is decaying in time as $\exp\left[-(1 + k^2)\alpha^2 t/R\right]$. It is readily seen that all the continuum modes are of the same general form in the far stream. The aspect ratio of the vortices and the distortion depend on k and $B(k)$.

Some related results have been obtained by Rogler (1975) and Rogler and Reshotko (1975) who considered the response of a boundary layer to an

array of nondecaying free stream vortices. This problem is not, as emphasized by Rogler and Reshotko, an eigenvalue problem, because the form of the vortex array is imposed in the far field, as well as the requirement that the vorticity be constant in time. Applying a perturbation analysis to the Navier-Stokes equations, Rogler and Reshotko obtain, formally, an inhomogeneous Orr-Sommerfeld equation with a forcing function depending on the mean velocity field and the vortex array. It should be emphasized that the vortex arrays considered by Rogler and Reshotko are not the continuum eigenfunctions of the Orr-Sommerfeld equation. The connection between these continuum modes and the free stream vortices used by Rogler and Reshotko will be discussed in a later paper (Part II).

3.2. The Two-dimensional (Plane) Laminar Jet

As a second example, the continuum modes of the two-dimensional laminar jet will be discussed. The velocity of this jet is, in dimensionless units,

$$U(y) = 1 - \tanh^2(y) . \tag{69}$$

The length scale, $L(x)$, and velocity scale, $U_0(x)$, are

$$L(x) = [48\,\nu^2\,x^2/K]^{1/3} , \tag{70}$$

and

$$U_0(x) = [3K^2/32\,\nu\,x]^{1/3} , \tag{71}$$

where, the cap ("^") denoting a dimensional quantity,

$$K = \int_{-\infty}^{\infty} \hat{U}(\hat{y})\,d\hat{y} \tag{72}$$

is the momentum flux per unit density, per unit width of the jet (Schlichting, 1951).

It is clear that the parallel shear flow approximation is poorer for a jet ($L(x) \sim x^{2/3}$) than for a boundary layer ($L(x) \sim x^{1/2}$), but we will not consider here the modification to the continuum necessary in the nonparallel flow approximation (but see below, section 4).

The free stream speed is zero for the jet, and so the continuous spectrum for the temporal stability problem lies along the line

$$c_T = - i\,(1 + k^2)\,\alpha/R\ . \tag{73}$$

As discussed above, this symmetric jet has two continuum modes, the symmetric and antisymmetric modes, for each point on the continuum. In both cases the modes are decaying, standing waves.

The continuum of the spatial stability problem for the jet lies along the curve

$$\alpha_r = (k\,R/2)\left\{\frac{1}{2}\left[\sqrt{1 + 16\,(\omega/k^2R)^2} - 1\right]\right\}^{1/2}, \tag{74}$$

$$\alpha_i = \omega k^{-1}\left\{\frac{1}{2}\left[\sqrt{1 + 16\,(\omega/k^2R)^2} - 1\right]\right\}^{-1/2}. \tag{75}$$

In contrast to the temporal stability problem, the continuum modes of the jet for the spatial stability problem are spatially decaying, traveling waves with a wave speed,

$$c = \omega/\alpha_r = (2\omega/k\,R)\left\{\frac{1}{2}\left[\sqrt{1 + 16\,(\omega/k^2R)^2} - 1\right]\right\}^{-1/2}. \tag{76}$$

We have calculated the symmetric and antisymmetric continuum modes of the laminar jet for the temporal stability problem at $\alpha = 1.0$, $R = 50.0$, and $0 < k \leq 100.0$, $(0° < \Theta \leq 89.4°)$. The numerical method is identical to that used with the boundary layer.

Although the form of the eigenfunction, $\phi(y)$, is different for the symmetric and antisymmetric modes, the complex amplitudes B and C at a given k are nearly identical. The amplitude and phase of B and C for both the symmetric and antisymmetric modes are plotted versus Θ in figure 5. The differences between the results for the symmetric and antisymmetric are so small that they cannot be shown on this figure.

The variations of the amplitudes and phases of B and C are much smoother for the jet than for the boundary layer. As $\Theta \to 0$, amp $(B) \to 1.0$ and amp $(C) \to 0$, while phase $(B) \to 180°$ and phase $(C) \to -45°$. The amplitudes of both B and C rise monotonically with increasing Θ to a peak response at $\Theta \simeq 79.5°$ $(k \simeq 5.15)$ and then decreases rapidly thereafter. The phases of B and C decrease monotonically with increasing Θ , slowly for $\Theta \lesssim 79.5°$ and then very rapidly. The phases are not shown for $\Theta > 85°$ because the extremely rapid variation with Θ in this region makes it impractical.

The maximum amplitudes of B and C are very large for the jet as compared to the boundary layer when it is noted that in both cases the normalization is the same, i.e., $A = 1$ for all Θ . In the case of the jet, the maximum amplitude of B is about 4850 (1.15 for a boundary layer), and the maximum amplitude of C is about 2300 (3.75 for a boundary layer). This was to be expected in view of the very low critical Reynolds number for a jet (of order 10) as compared to the critical Reynolds number for the Blasius boundary layer (about 300 with this length scale).

This same sensitivity of the jet to the free stream disturbances can be seen in figures 6 and 7 where the Re (ϕ) and Im (ϕ) are plotted versus y for the symmetric and antisymmetric modes with $\alpha = 1.0$, $R = 50.0$, and $k = 2.0$ ($\theta = 63.43°$). It can be seen that in both cases the disturbance penetrates much further into the jet than was the case for the boundary layer. If the "edge" of the undisturbed jet is defined as the point where $U(y) \simeq 0.001$, then the jet occupies the region $|y| \leq 3$, and it is seen from figures 6 and 7 that there is quite a substantial disturbance in this region. If the amplitude of the disturbance were large enough the apparent "edge" of the jet could be moved by the disturbance.

Figure 8 shows the far field stream function for the symmetric mode of the jet at $\alpha = 1.0$, $R = 50.0$, and $k = 2.0$. The region shown is $0 \leq \alpha x \leq 8\Pi$ and $10\Pi \leq \alpha y \leq 14\Pi$. As in the case of the Blasius boundary layer, the far field is a doubly periodic array of vortices, but in contrast to the boundary layer the vortices are highly distorted due to the fact that $|B|$ is considerably different from $|A| = 1$. At this distance, $y > 30$, from the jet axis the trapped mode is negligible. The far field stream function for the antisymmetric mode is essentially identical to that of the symmetric mode and is not shown.

4. THE CONTINUUM MODES AND TRANSITION

All of the continuum modes, both temporal and spatial, for the linearized stability problem of an unbounded parallel shear flow are damped; the flow is linearly stable with respect to these modes. It is, of course, possible that if the free stream vorticity is sufficiently intense the nonlinear problem would yield growing solutions, i.e., the flow would be unstable to these modes and transition would result.

We believe, however, that there are three possible mechanisms by which <u>small amplitude</u> continuum modes might lead to transition. The first mechanism involves a small quasi-steady distortion of the mean profile by the continuum modes; the second mechanism requires the consideration of the fully time-dependent flow, consisting of the mean flow plus the slowly decaying continuum modes; while the third mechanism involves consideration of the nonparallel flow corrections to the continuum spectrum.

The usual picture of transition via T-S waves in a boundary layer consists of: the appearance of a small amplitude, unstable T-S wave of unspecified origin; its growth through the region of validity of the linear theory; the appearance of small nonlinear effects, i.e., higher harmonics and, more importantly, a small three-dimensional distortion of the mean profile near the top of the boundary layer; and finally the appearance of short wavelength, rapidly growing T-S waves which lead directly to the appearance of a turbulent burst. It is believed that the penultimate stage, the distortion of the mean profile near the top of the boundary layer, is crucial. This distortion, in the region where U'' is small, causes U'' to change sign, and the inflection in the mean profile permits the appearance of small wavelength, highly unstable T-S waves. These, presumably, become nonlinear rather quickly and cause a further distortion of the mean profile, and so on to transition. This is, in effect, a modified Landau model.

Now consider the effect of a small patch of vorticity in the free stream. Assuming, as we will show in Part II, that we can represent this by an integral over the continuum, all wavelength continuum modes are present and exciting the boundary layer. Among these are small α, small k continuum modes, i.e., lightly damped, long wavelength and, hence, low frequency modes.

In a quasi-steady model these produce a small, quasi-steady distortion of the mean profile in the outer regions where U'' is small; therefore, we might expect that the boundary layer has reached the critical stage when short wavelength, highly unstable T-S waves can appear without ever experiencing a long wavelength, slowly growing T-S wave. In effect, this mechanism would bypass the linear T-S stage and appear to lead directly to the late stages of transition.

The second mechanism involves consideration of the fully time-dependent flow, consisting of a superposition of the mean flow and the small amplitude continuum modes. Again consider the effect of a small patch of free stream vorticity. Among the modes necessary to represent this patch there will be small α modes, i.e., long wavelength, lightly damped waves. Let us consider the stability of this time-dependent flow: the mean flow plus the lightly damped modes. Relatively little is known about the stability of time-dependent shear flows (see Davis, 1976 for a recent review), but it is known that under certain circumstances these flows can have resonances. That is, if the "matching" is proper, the time-dependent flow can be unstable to T-S like modes which extract energy from the mean flow at a rapid rate; they are highly unstable. The unstable mode tends to lie near a higher mode of the steady Orr-Sommerfeld equation (Grosch and Salwen, 1969). Again, if this were to occur, the linear T-S region of instability would be bypassed and it would appear, in an experiment, that the short wavelength, highly unstable waves had been directly excited.

Finally, let us consider the possible effects of nonparallel flow corrections on these continuum modes. Saric and Nayfeh (1975) have developed a general theory which permits the calculation of the correction to the

spectrum of the Orr-Sommerfeld equation due to the nonparallel nature of any nearly parallel shear flow. They have applied this theory to the stability of the Falkner-Skan flows. In all cases which they considered, the nonparallel flow corrections resulted in a decrease in the stability of the boundary layer as compared to the parallel flow calculation at fixed (real) wave number α and Reynolds number R. That is, at a fixed α, the flow was predicted to be unstable at a lower R than that predicted by the parallel shear flow approximation; and, at a fixed R, the band of unstable wave numbers was wider than that predicted from the usual parallel flow approximation.

The continuum modes, particularly at the long wavelength end of the spectrum, are very lightly damped. It does not appear to be unreasonable that nonparallel flow corrections may make a portion of the continuous spectrum unstable.

All of these mechanisms are, at this time, only hypotheses. We know of no direct evidence for or against any of them, other than the observations that free stream turbulence can trigger transition without the appearance of the usual T-S waves. We believe, however, that all of these mechanisms are sufficiently plausible that experimental and theoretical tests of them is warranted. We hope to carry out a theoretical investigation of these proposed transition mechanisms; we are currently investigating the nonparallel flow corrections.

5. SUMMARY

We have shown that the Orr-Sommerfeld equation (for both the temporal and spatial problems) has a continuous spectrum for any mean flow which is an

unbounded shear flow and has finite energy under some Galilean transformation.

Formulae for the location of the continuum in the complex wave speed plane have been given. These results have been applied to two specific flows: the Blasius boundary layer and the laminar jet. For both of these flows the continuous spectra have been given. The continuum eigenfunctions of the temporal problem have been calculated for both flows for a single wave number and Reynolds number. It has been shown that the eigenfunctions in the free stream are a doubly periodic array of vortices and that these eigenfunctions can be thought of as an incident wave, a reflected wave and a "wall" wave. Holding the amplitude of the incident wave fixed, the amplitudes and phases of the reflected and "wall" waves vary with the angle of incidence, and there is an optimum angle of incidence which maximizes the amplitude of the wall wave. It was shown that the continuum modes do not penetrate deeply into the region of large shear in the main flow of a boundary layer although they penetrate further and have much larger amplitudes in the jet than in the boundary layer. Finally, three mechanisms were proposed by which these continuum modes could cause transition in a shear flow while bypassing the usual linear T-S stage. These mechanisms are, at present, hypotheses; we hope to investigate these hypotheses theoretically as well as to investigate the relation of these eigenfunctions to the forced free stream disturbances studied by others. We hope that experimentalists will also look for these modes.

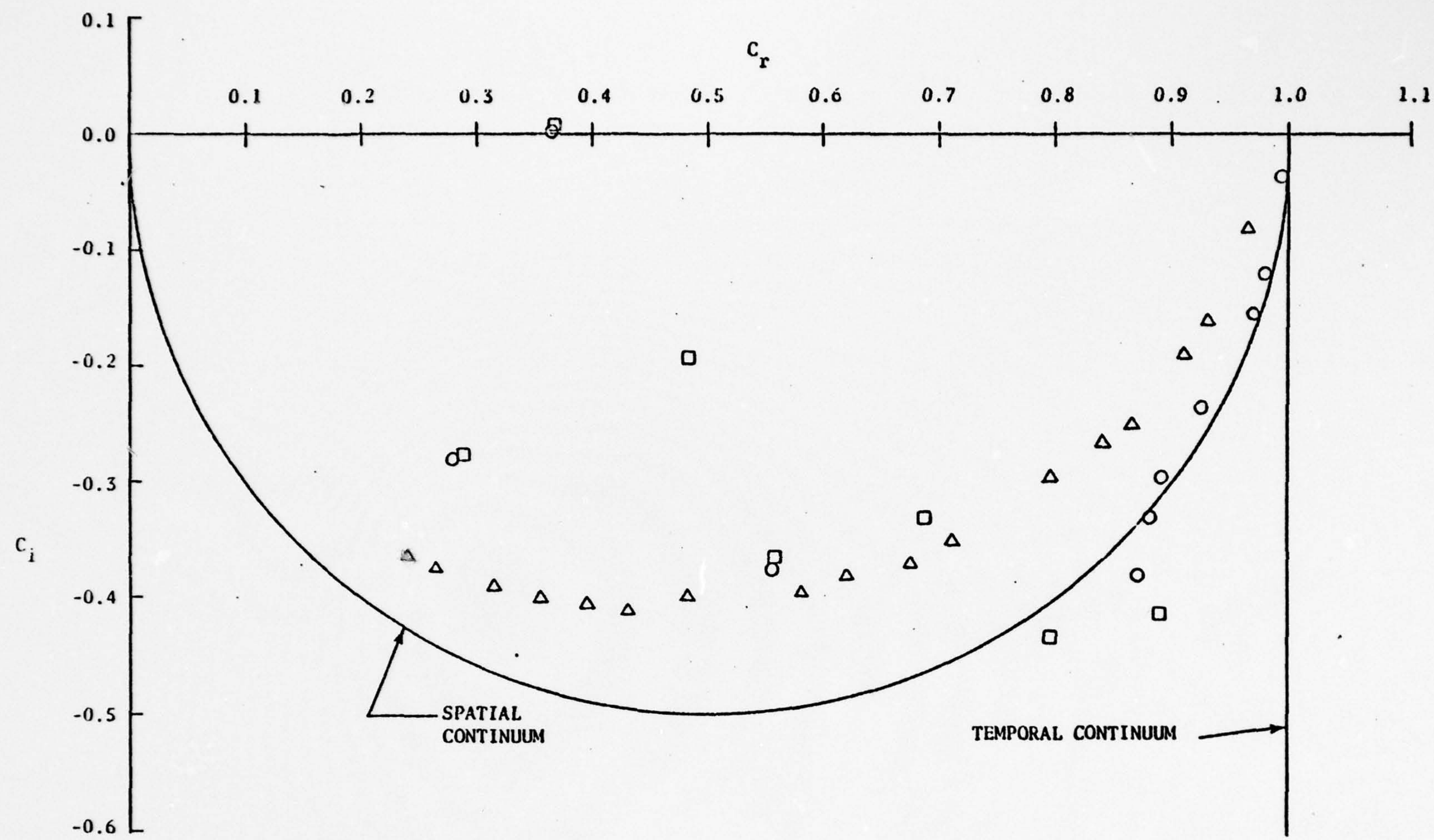

Figure 1. The spatial and temporal spectrum of the Orr-Sommerfeld equation for Blasius flow at $\dot{R} = 580.0$ and $\omega = 0.0652$ (spatial) or $\alpha = 0.179$ (temporal). The points of the discrete spectrum at this R and ω or α, as computed by Jordinson (1971), are denoted by O-temporal modes, and Δ-spatial modes, while the results of Mack's calculations (temporal modes) are denoted by □ .

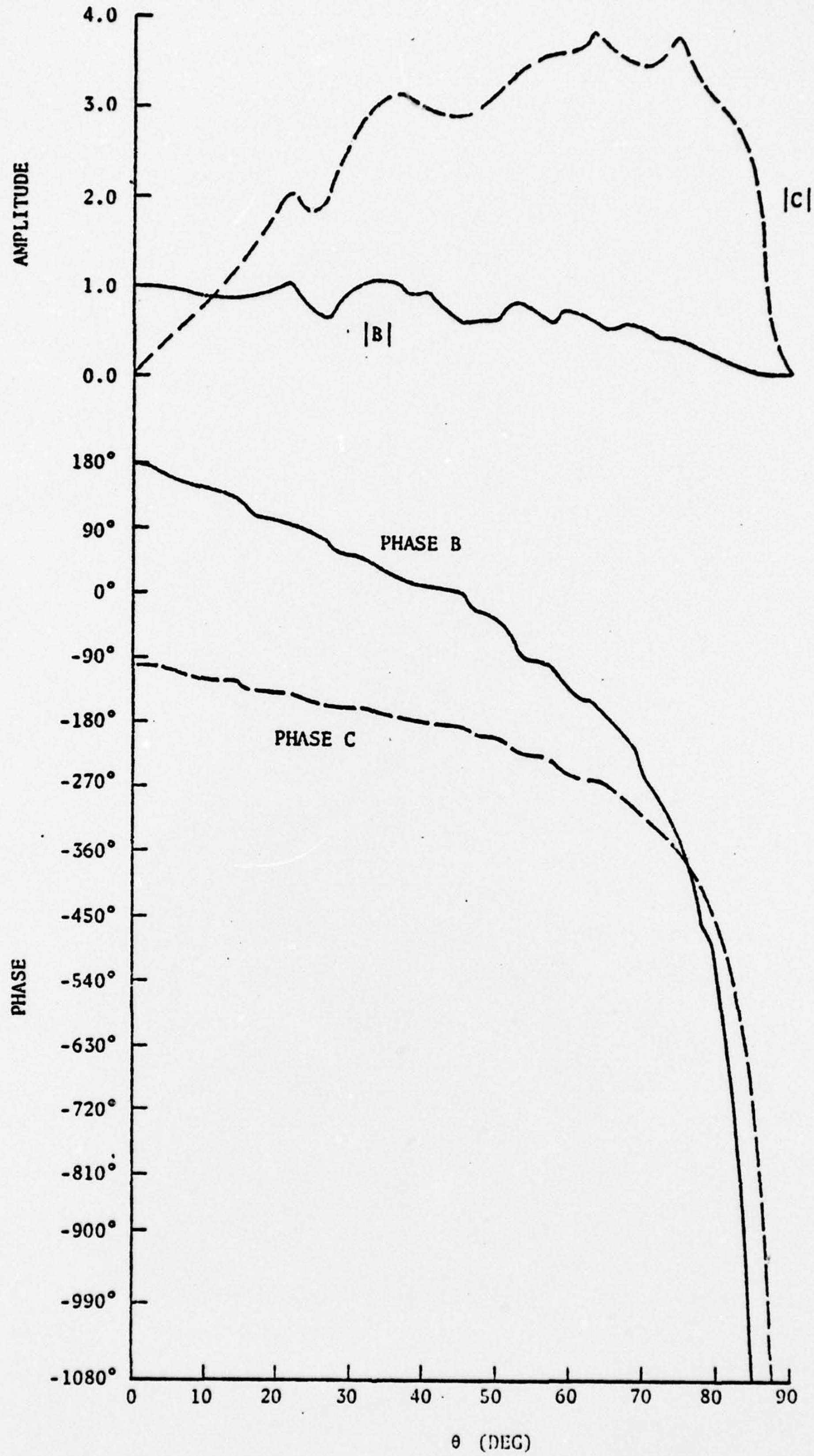

Figure 2. The magnitudes and phases of the complex amplitudes B and C for the continuum modes of the Blasius boundary layer at α = 0.179, R = 580.

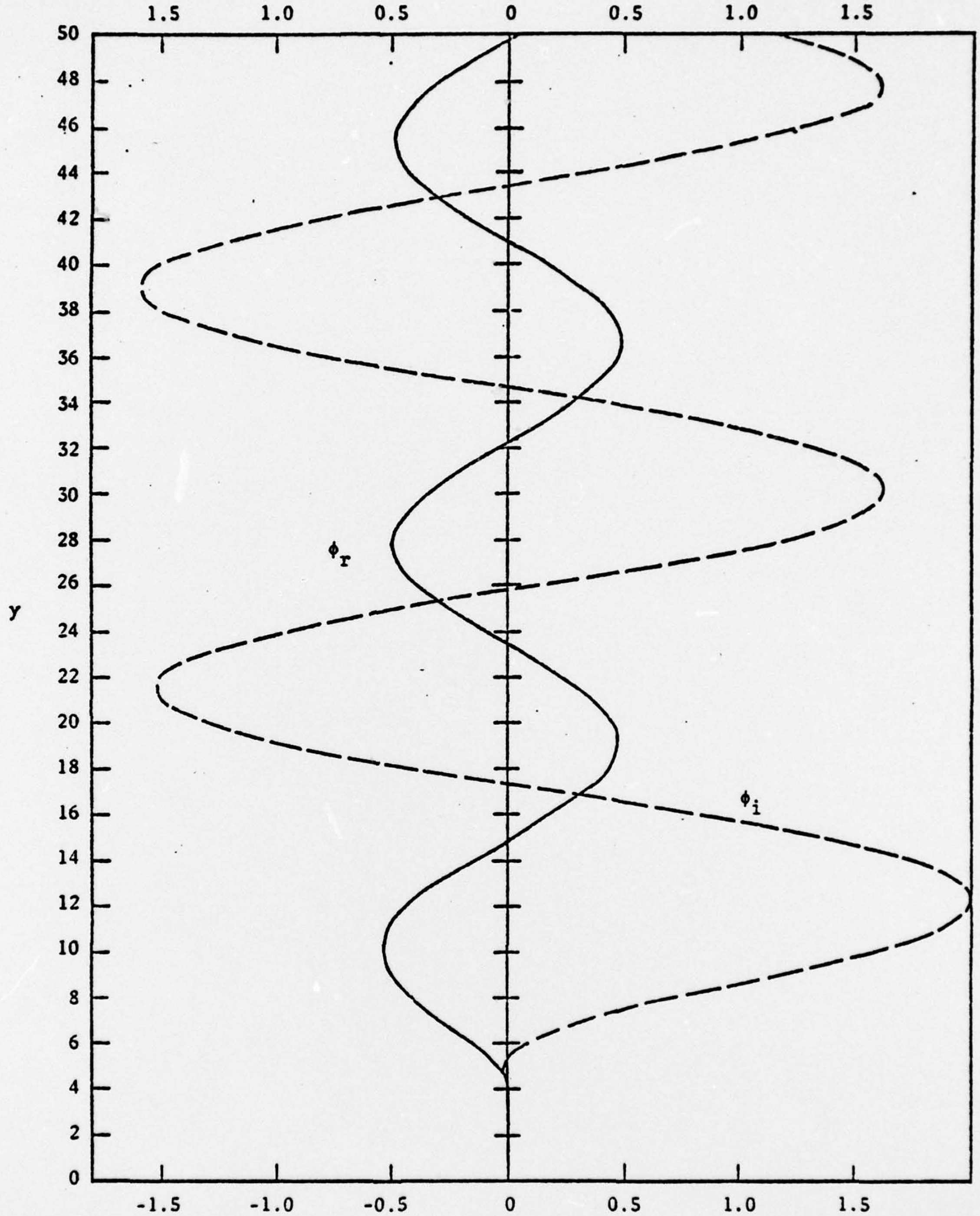

Figure 3. The real $\phi_r(y)$ (solid line) and imaginary $\phi_i(y)$ (dashed line), parts of $\phi(y)$ versus y of the continuum mode of the Blasius boundary layer at $\alpha = 0.179$, $R = 580$, and $k = 2$. ($\Theta = 63.43°$).

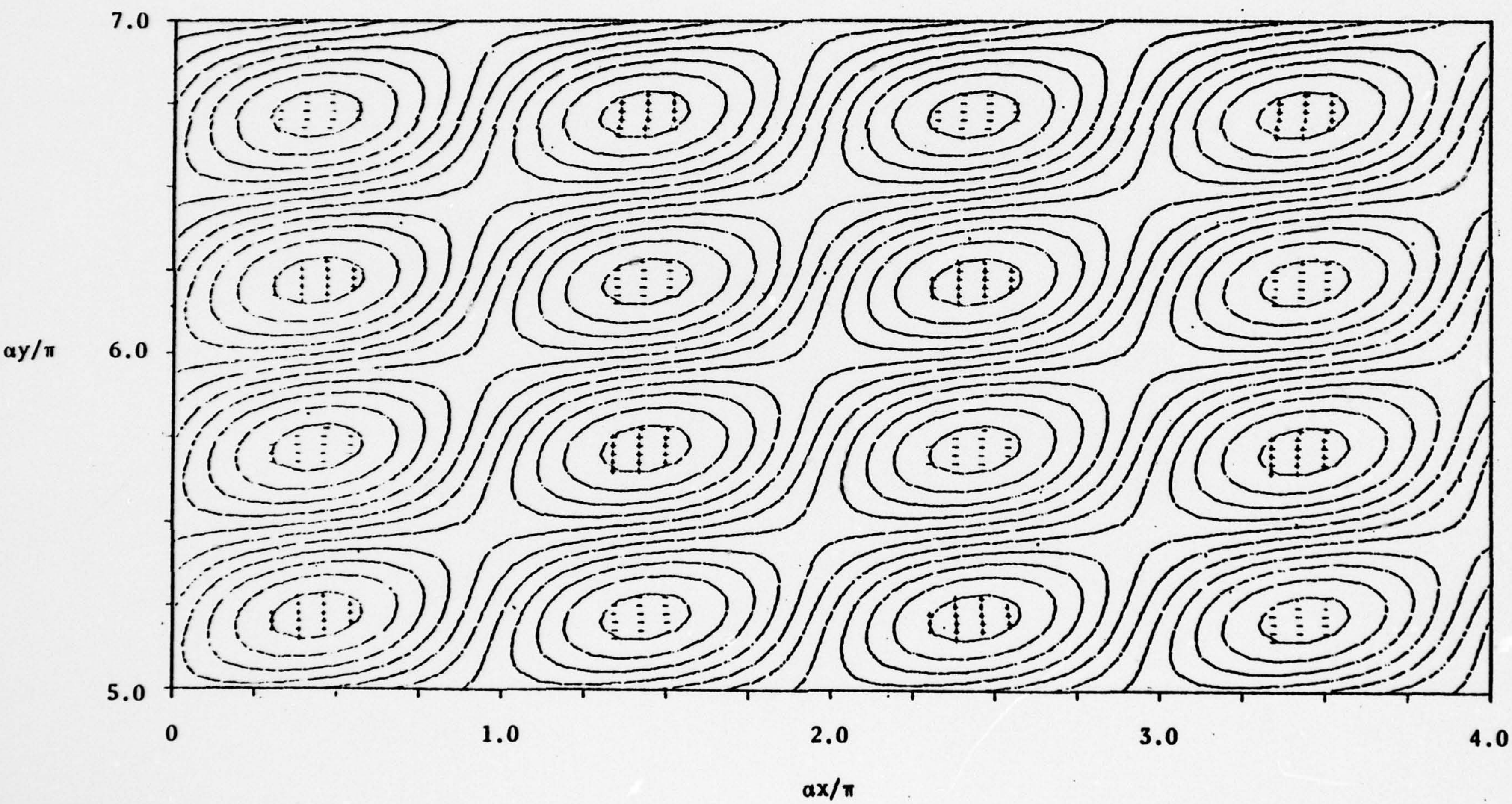

Figure 4. The far field stream function of the continuum mode of the Blasius boundary layer at $\alpha = 0.179$, $R \doteq 580$, and $k = 2$. ($\Theta = 63.43°$).

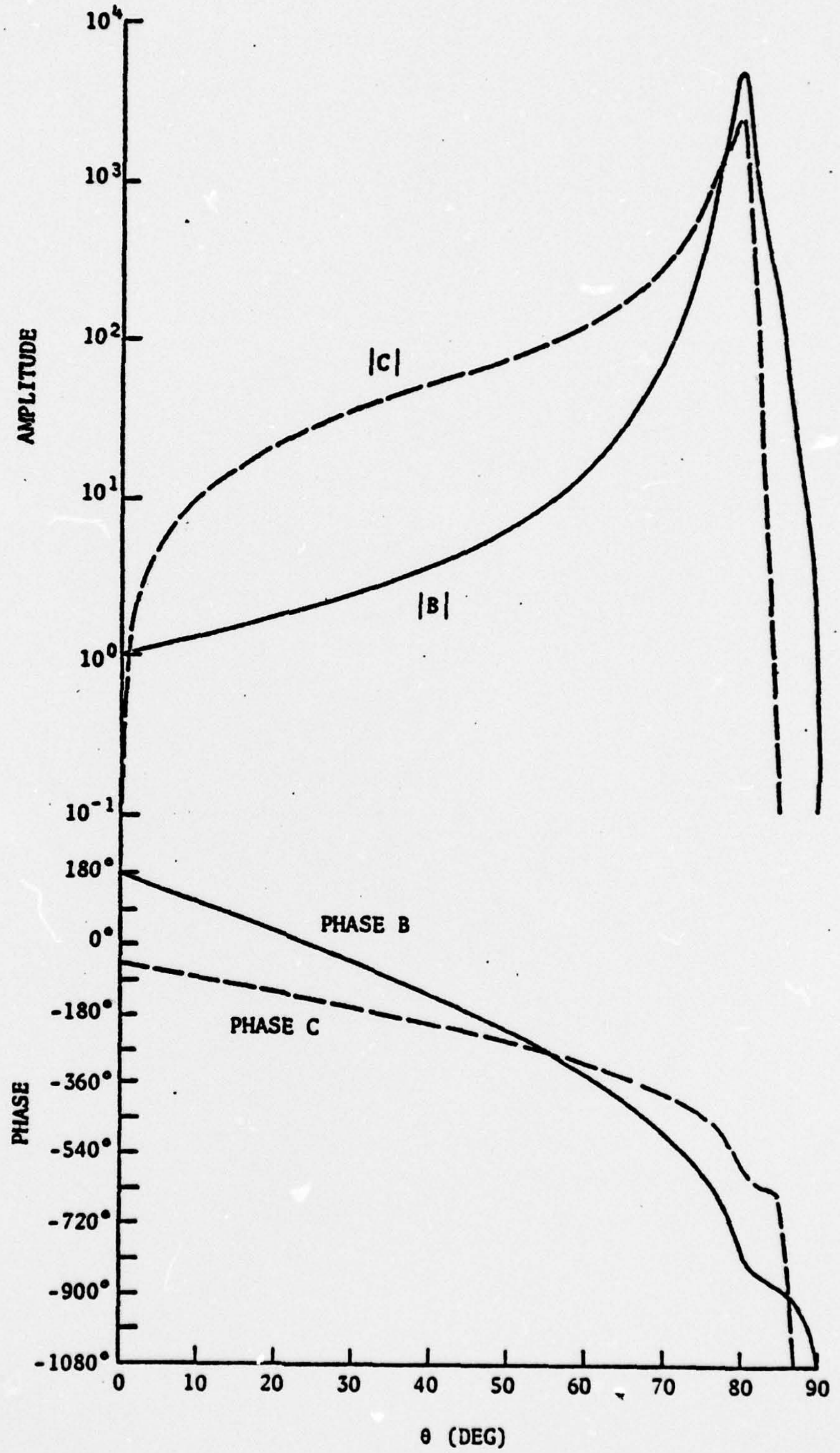

Figure 5. The magnitude and phase of the complex amplitudes B and C for the continuum modes of the two-dimensional laminar jet at $\alpha = 1$ and $R = 50$.

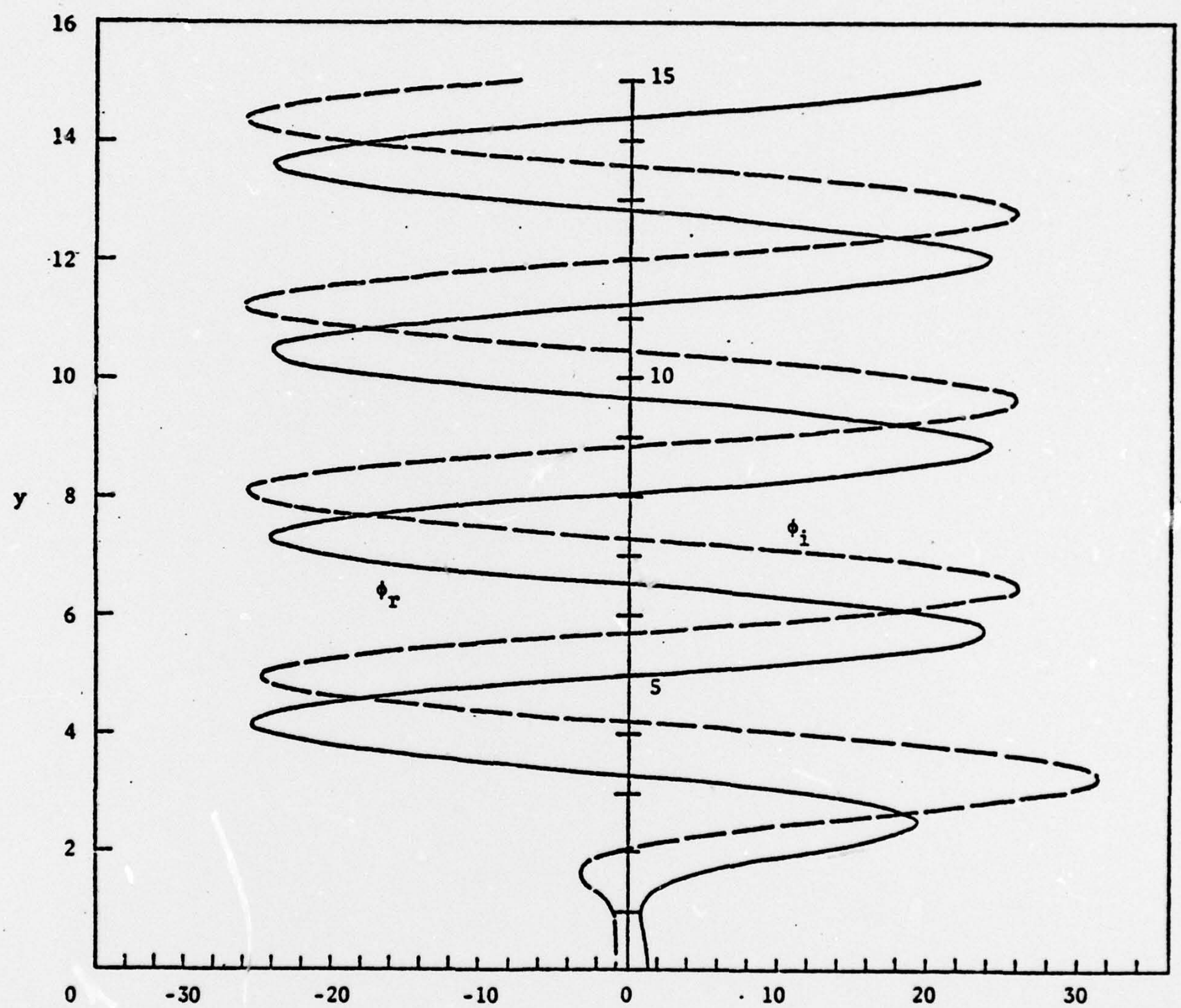

Figure 6. The real ϕ_r (y) (solid line) and the imaginary ϕ_i (y) (dashed line), parts of ϕ for the symmetric temporal stability mode of a two-dimensional laminar jet. Here $\alpha = 1$, $R = 50$, and $k = 2$. $(\Theta = 63.43°)$.

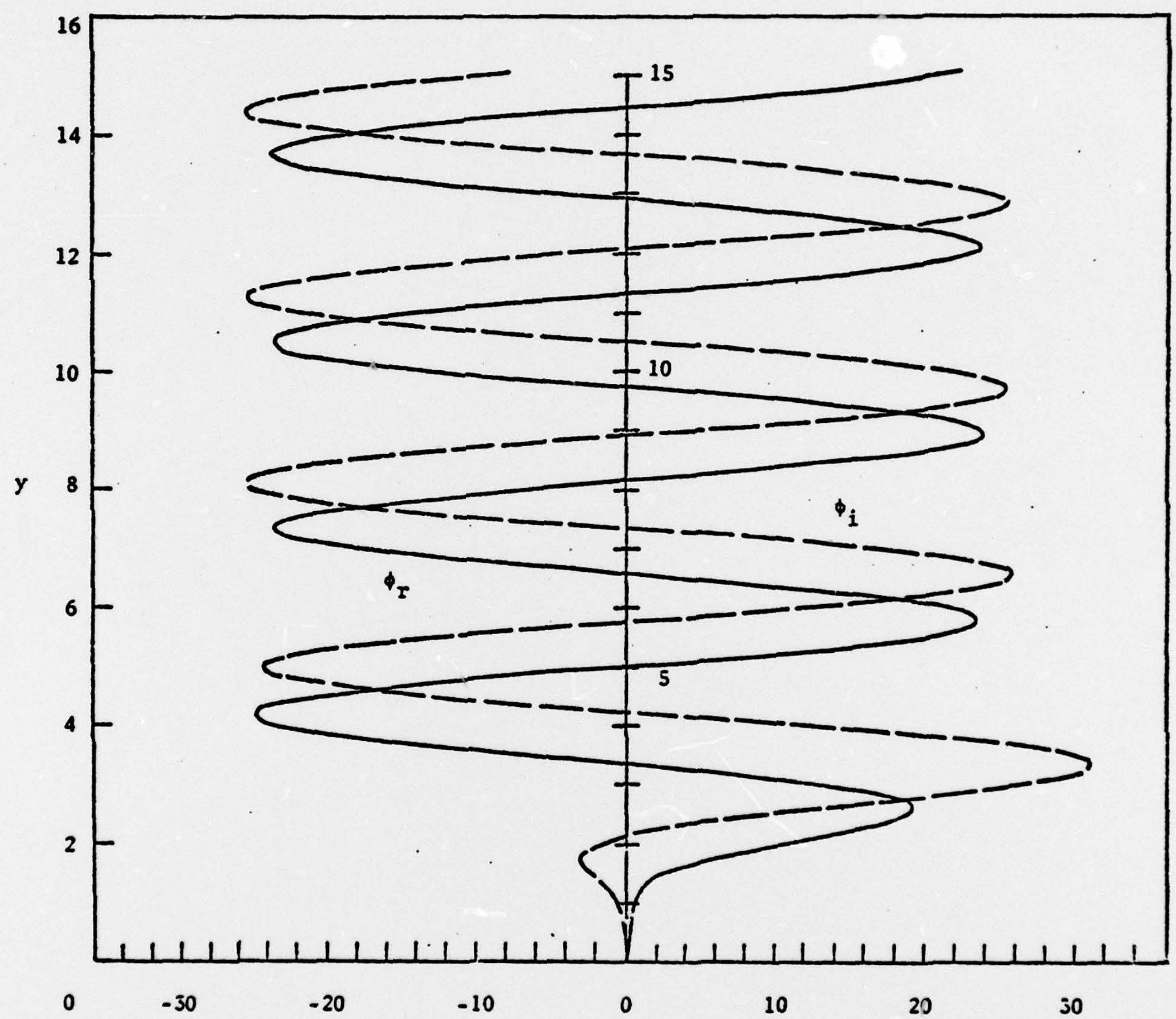

Figure 7. The real ϕ_r (y) (solid line) and the imaginary ϕ_i (y) (dashed line), parts of ϕ for the antisymmetric temporal stability mode of a two-dimensional laminar jet. Here $\alpha = 1$, $R = 50$, and $k = 2$. ($\Theta = 63.43°$).

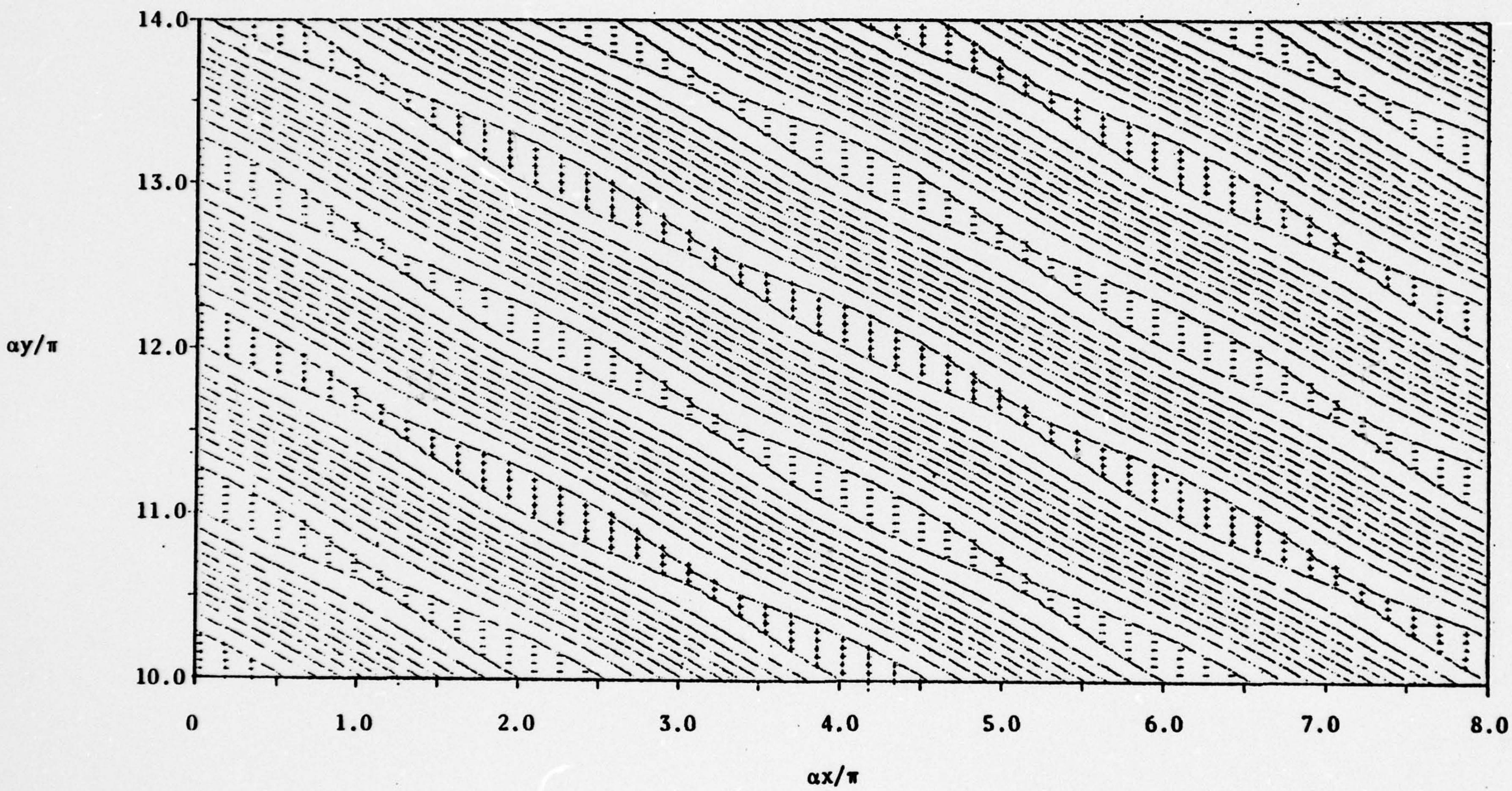

Figure 8. The far field stream function of the symmetric continuum mode of the laminar jet at $\alpha = 1$, $R = 50$, and $k = 2$. $(\theta = 63.43°)$.

REFERENCES

Batchelor, G. K., 1967. An Introduction to Fluid Dynamics, Cambridge: Cambridge University Press.

Bethe, H. A., 1947. Elementary Nuclear Theory, New York: John Wiley & Sons, Inc.

Case, K. M., 1960. "Stability of inviscid Couette flow," Phys. Fluids, 3, 143.

Case, K. M., 1961. "Hydrodynamic stability and the inviscid limit," J. Fluid Mech., 10, 420.

Davis, S. H., 1976. "The stability of time-periodic flows," in Annual Rev. Fluid Mechanics, M. Van Dyke, W. G. Vincenti, and J. V. Wehausen, eds., Palo Alto: Annual Rev. Inc.

Di Prima, R. C. and Habelter, G. J., 1969. "A completeness theorem for nonself-adjoint eigenvalue problems in hydrodynamic stability," Arch. Rat. Mech. & Anal., 34, 218.

Friedman, B., 1956. Principles and Techniques of Applied Mathematics, New York: John Wiley & Sons, Inc.

Gallagher, A. P. and Mercer, A. McD., 1964. On the behavior of small disturbances in plane Couette flow; Part 2, The higher eigenvalues, J. Fluid Mech. 18, 350.

Grosch, C. E. and Salwen, H., 1968. "The stability of steady and time-dependent plane Poiseuille flow," J. Fluid Mech., 34, 177.

Haupt, O., 1912. "Über die Entwicklung einer willkürlichen Funktion nach den Eigenfunktionen des Turbulenzproblems," Sber. bayer. Akad. Wiss., 2, 289.

Howard, L. N., 1961. "Note on a paper of John W. Miles," J. Fluid Mech., 10 509.

Jordinson, R., 1971. "Spectrum of eigenvalues of the Orr-Sommerfeld equation for Blasius flow," Phys. Fluids, 14, 177.

Kaplan, R. E., 1964. "The stability of laminar incompressible, boundary layers in the presence of compliant boundaries," Mass. Inst. Tech. ASRL TR 116-1.

Landahl, M., 1966. "A time-shared program for the stability problem for parallel flows over rigid or flexible surfaces," Mass. Inst. Tech., ASRL TR 116-4.

Landau, L. D. and Lifshitz, E. M., 1959. Quantum Mechanics, Reading, Mass.: Addison-Wesley Pub. Co.

Lessen, M. S., Sadler, G. and Liu, T. Y., 1968. "Stability of pipe Poiseuille flow," Phys. Fluids, 11, 1404.

Lin, C. C., 1955. The Theory of Hydrodynamic Stability, Cambridge: Cambridge Univ. Press.

Lin, C. C., 1961. "Some mathematical problems in the theory of the stability of parallel flows," J. Fluid Mech., 10, 430.

Mack, L. M., 1976. "A numerical study of the temporal eigenvalue spectrum of the Blasius boundary layer," to appear in J. Fluid Mech.

Obremski, H. J., Morkovin, M. J. and Landahl, M., 1969. "A portfolio of stability characteristics of incompressible boundary layers," AGARDograph 134.

Orszag, S. A., 1971. "Accurate solution of the Orr-Sommerfeld stability equation," J. Fluid Mech., 50, 689.

Radbill, J. R. and Van Driest, E. R., 1966. "A new method for the prediction of stability of laminar boundary layers," North American Aviation Report AD 633 978.

Rayleigh, 1880. "On the stability, or instability, of certain fluid motions," Sci. Papers, 1, 474, Cambridge Univ. Press.

Reynolds, O., 1883. "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and the law of resistance in parallel channels," Phil. Trans. Roy. Soc. London, 174, 935.

Rogler, H. L., 1975. "The interaction between vortex-array representations of freestream turbulence and impermeable bodies," AIAA 13th Aerospace Sciences Meeting, Pasadena, Cal.

Rogler, H. L. and Reshotko, E., 1975. "Disturbances in a boundary layer introduced by a low intensity array of vortices," SIAM. J. Appl. Math., 28, 431.

Rosencrans, S. A. and Sattinger, D. H., 1966. "On the spectrum of an operator occurring in the theory of hydrodynamic stability," J. Math. and Phys., 45, 289.

Salwen, H. and Grosch, C. E., 1972. "Stability of Poiseuille flow in a circular pipe," J. Fluid Mech., 54, 93.

Saric, W. S. and Nayfeh, A. H., 1975. "Nonparallel stability of boundary-layer flows," Phys. Fluids, 18, 945.

Sattinger, D. H., 1967. "On the Rayleigh problem in hydrodynamic stability," SIAM. J. Appl. Math., 15, 419.

Schensted, I. V., 1960. "Contributions to the theory of hydrodynamic stability," Ph.D. dissertation, University of Michigan.

Schlichting, H., 1951. Boundary Layer Theory, New York: McGraw-Hill Book Company.

Schubauer, G. B. and Skramstad, H. K., 1941. "Laminar boundary-layer oscillations and stability of laminar flow," NACA Rept. W8.

Schubauer, G. B. and Skramstad, H. K., 1947. "Laminar boundary-layer oscillations and stability of laminar flow," J. Aero. Sci., 14, 69.

Schubauer, G. B. and Skramstad, H. K., 1948. "Laminar boundary-layer oscillations and stability of laminar flow," NACA Rept. 909.

Squire, H. B., 1933. "On the stability of the three dimensional disturbances of viscous flow between parallel walls," Proc. Roy. Soc. London A142, 621.

Tollmien, W., 1931. "Uber die entstehung der turbulenz," NACA Tech. Memo. No. 609.

Wazzan, A. R., Okamura, T. T. and Smith, A. M. O., 1968. "Spatial and temporal stability charts for the Falkner-Skan boundary layer profiles," Douglas Aircraft Company, Rept. No. DAC-67086.